The Paradox of Idealization *

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A well-known proof by Alonzo Church, first published in 1963 by Frederic Fitch, shows that all truths are knowable only if all truths are known.[1] This is the Paradox of Knowability. If we take it, quite plausibly, that we are not omniscient, the proof appears to undermine metaphysical doctrines committed to the knowability of truth, such as semantic anti-realism. Since its rediscovery by Colin McGinn and William Hart in 1976,[2] many solutions to the Paradox have been offered. In this paper, we show that some of them do not have the resources to block a problem we raise. We present a new proof to the effect that not all truths are knowable, resting on different assumptions from the original argument published by Fitch. In light of this proof, anti-realists who favour either a hierarchical or an intuitionistic approach to the Paradox of Knowability are confronted with a dilemma: they must either give up anti-realism or opt for a highly controversial interpretation of their original tenet.

1 The Church-Fitch Paradox

The proof of the Church-Fitch Paradox requires only that knowledge be factive and that it distribute over conjunction. In symbols:

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[1]See Fitch (1963) and Church (forthcoming).

[2]See Hart and McGinn (1976).

$$(\mathsf{FACT})\ \frac{\mathcal{K}\varphi}{\varphi} \qquad (\mathsf{DIST})\ \frac{\mathcal{K}(\varphi \wedge \psi)}{\mathcal{K}\varphi \wedge \mathcal{K}\psi},$$

where $\mathcal{K}\varphi$ reads 'someone knows at some time that φ'. Let '$\Diamond$' and '$\Box$' denote some notion of possibility and some correlative notion of necessity respectively. Then, we can prove that what Williamson (2000) calls *weak verificationism*:

$$(\mathsf{WVER})\ \ \forall\varphi(\varphi \to \Diamond\mathcal{K}\varphi),$$ [3]

collapses into *strong verificationism*:

$$(\mathsf{SVER})\ \ \forall\varphi(\varphi \to \mathcal{K}\varphi).$$

The proof is in two stages. We first show that for any particular proposition p, $\neg\Diamond\mathcal{K}(p \wedge \neg\mathcal{K}p)$ is provable:

1	(1)	$\mathcal{K}(p \wedge \neg\mathcal{K}p)$	A
1	(2)	$\mathcal{K}p \wedge \mathcal{K}\neg\mathcal{K}p$	1, DIST
1	(3)	$\mathcal{K}p \wedge \neg\mathcal{K}p$	2, FACT
	(4)	$\neg\mathcal{K}(p \wedge \neg\mathcal{K}p)$	1–3, $\neg$-I
	(5)	$\Box\neg\mathcal{K}(p \wedge \neg\mathcal{K}p)$	4, Necessitation
	(6)	$\neg\Diamond\mathcal{K}(p \wedge \neg\mathcal{K}p)$	5, $\Box\neg\phi \to \neg\Diamond\phi$

We then proceed to show that, given (6), all truths are knowable only if all truths are known:

7	(7)	$\forall\varphi(\varphi \to \Diamond\mathcal{K}\varphi)$	A
7	(8)	$(p \wedge \neg\mathcal{K}p) \to \Diamond\mathcal{K}(p \wedge \neg\mathcal{K}p)$	7, $\forall$-E
9	(9)	$p \wedge \neg\mathcal{K}p$	A
7, 9	(10)	$\Diamond\mathcal{K}(p \wedge \neg\mathcal{K}p)$	8, 9, $\to$-E
7, 9	(11)	$\Diamond\mathcal{K}(p \wedge \neg\mathcal{K}p) \wedge \neg\Diamond\mathcal{K}(p \wedge \neg\mathcal{K}p)$	10, 6, $\wedge$-I
7	(12)	$\neg(p \wedge \neg\mathcal{K}p)$	9–11, $\neg$-I
7	(13)	$p \to \mathcal{K}p$	12, PC (Classical)
7	(14)	$\forall\varphi(\varphi \to \mathcal{K}\varphi)$	13, $\forall$-I
	(15)	$\forall\varphi(\varphi \to \Diamond\mathcal{K}p) \to \forall\varphi(\varphi \to \mathcal{K}\varphi)$	7-14, $\to$-I

In a nutshell, the Paradox shows that, if $\mathcal{K}$ is factive and distributes over conjunction, truths of the form $p \wedge \neg\mathcal{K}p$ are provably unknowable. Yet on the

[3] The principle is usually meant to apply only to those propositions expressed by sentences we do understand, and quantifiers are usually interpreted substitutionally. We set aside these complications for present purposes.

anti-realist assumption that all truths are knowable, unknowable propositions are to be regarded as false. By an elementary, though exclusively classical step, it follows that $\forall\varphi(\varphi \rightarrow \mathcal{K}\varphi)$. Since this latter claim is clearly false—indeed, we are not omniscient—anti-realism is under threat.

2 Intuitionistic Ways Out

According to Timothy Williamson (1982), the Knowability Paradox is no straightforward *reductio* of semantic anti-realism. As he points out, the step from (12) to (13) is invalid within the anti-realist's preferred logic, *viz.* intuitionistic logic. Intuitionistically, (12) only implies:

$$(\mathsf{WVER}^*)\ \forall\varphi(\varphi \rightarrow \neg\neg\mathcal{K}\varphi).$$

But unlike SVER, WVER* is not obviously problematic. As Williamson puts it:

> it forbids intuitionists to produce claimed *instances* of truths that will never be known: but why should they attempt something so foolish? (Williamson, 1982, p. 206)

Given the intuitionistic invalidity of the step from $\neg\forall x\phi$ to $\exists x\neg\phi$, Williamson further points out, intuitionists can even consistently negate that all truths will be known at some time without thereby being committed to the existence of a forever unknown truth. Hence, they can posit a logical gap between WVER and SVER. The Paradox, Williamson argues, constrains anti-realism but does not obviously undermine it:

> That a little logic should short circuit an intensely difficult and obscure issue was perhaps too much to hope, or fear (Williamson, 1982, p. 207).

It is however unclear whether the adoption of intuitionistic logic may itself constitute a fully satisfactory solution of the problem raised by Church-Fitch. Although WVER* might be acceptable, *other* intuitionistic consequences of WVER already seem worrisome. Intuitionists, for instance, are still committed to

$$(12')\ \forall\varphi\neg(\varphi \wedge \neg\mathcal{K}\varphi),$$

which is tantamount to denying the highly plausible claim that there exist forever unknown truths.[4] Wouldn't this show that reverting to intuitionistic logic is a no use strategy for the anti-realist?

Michael Dummett has recently responded that intuitionist anti-realists are not committed to the existence of forever unknown truths in the first place.[5] The standard argument for $\exists\varphi(\varphi \wedge \neg\mathcal{K}\varphi)$ runs thus. Consider some decidable statement p such that all the evidence for or against it has been lost—say "The number of hairs now on Dummett's head is even", as uttered just before some of Dummett's hairs are burned. Given that p is decidable—we could have counted Dummett's hairs—even intuitionists should be willing to assert that either it or its negation is true. Since however both p and $\neg p$ are *ex hypothesis* forever unknown, the disjunction $(p \wedge \neg\mathcal{K}p) \vee (\neg p \wedge \neg\mathcal{K}\neg p)$ also holds, from which $\exists\varphi(\varphi \wedge \neg\mathcal{K}\varphi)$ trivially follows.[6]

What, if anything, has gone wrong? The problem with this reasoning, Dummett claims, is that, for the anti-realist, the truth of p and $\neg p$ amounts to the truth of the counterfactual conditionals "If we had counted Dummett's hairs, they would have proved to be even in number" and "If we had counted Dummett's hairs, they would have proved to be odd in number" respectively.[7] But that one of these two counterfactuals is true, he says,

> cannot be inferred from the unquestionable truth that, if [Dummett's] hairs were counted, they would be found to be either even or odd in number. (Dummett, 2007b, p. 350)

This would be an instance of the problematic step from $\phi \;\Box\!\!\rightarrow (\psi \vee \chi)$ to $(\phi \;\Box\!\!\rightarrow \psi) \vee (\phi \;\Box\!\!\rightarrow \chi)$.[8] And this inference, Dummett writes, "does follow in the

[4]Besides (12′), Philip Percival points out two more untoward intuitionistic consequences of WVER: $\forall\varphi(\neg\mathcal{K}\varphi \leftrightarrow \neg\varphi)$ and $\forall\varphi\neg(\neg\mathcal{K}\varphi \wedge \neg\mathcal{K}\neg\varphi)$. See Percival (1990).

[5]See Dummett (2007b, p. 348–50). Notice that Dummett's new take on the Paradox is in stark contrast with the solution proposed in Dummett (2001). That paper, he now writes, was written "in a mood of irritation with the paradox of knowability" (Dummett, 2007b, p. 348).

[6]*Proof*: Assume $(p \wedge \neg\mathcal{K}p) \vee (\neg p \wedge \neg\mathcal{K}\neg p)$ for disjunction elimination. Then assume $(p \wedge \neg\mathcal{K}p)$ for arrow introduction. By existential introduction, derive $\exists\varphi(\varphi \wedge \neg\mathcal{K}\varphi)$. By arrow introduction, it follows that $(p \wedge \neg\mathcal{K}p) \rightarrow \exists\varphi(\varphi \wedge \neg\mathcal{K}\varphi)$. By similar reasoning, show that $(\neg p \wedge \neg\mathcal{K}\neg p) \rightarrow \exists\varphi(\varphi \wedge \neg\mathcal{K}\varphi)$. By disjunction elimination, it follows that $\exists\varphi(\varphi \wedge \neg\mathcal{K}\varphi)$. ■

[7]See Dummett (2007b, p. 349).

[8]Here is the countermodel. Consider a model consisting of three worlds, w_1, w_2 and w_3, such that $w_2 \models p \wedge q$ and $w_3 \models p \wedge r$. Now let w_2 and w_3 be the closest p-worlds to

mathematical case, but not in the empirical case [...], the reason [being] that the outcome of the mathematical procedure is determined entirely internally, but that of the empirical procedure is not" (Dummett, 2007b, p. 349).[9] For empirical p's, the thought is, we have no guarantee that we will always be in a position to apply our decision procedure: the right time could pass, like in the example above.[10] In order to assume bivalence for empirical statements that *could have been known*, but whose knowledge is now beyond our ken, the *unrestricted* law of bivalence is needed, or so Dummett argues:

> [the realist] relies on assuming bivalence in order to provide an example of a true statement that will never be known to be true—more exactly, of a pair of statements one of which is true. He has to. If he could instance a specific true statement, he would know that it was true. This illustrates how important the principle of bivalence is in the controversy between supporters and opponents of realism. (Dummett, 2007b, p. 350)

Now recall the derivation of the Paradox of Knowability: the argument's bulk shows that WVER is inconsistent with the existence of truths of the form $\varphi \wedge \neg\mathcal{K}\varphi$. If the latter assumption is intuitionistically unacceptable, though, there might well be no inconsistency to be had.

3 Typing Knowledge

A second, quite natural way to block the Paradox had already been suggested by Church in 1945:

> Of course the foregoing refutation [...] is strongly suggestive of the paradox of the liar and other epistemological paradoxes. It may therefore be that Fitch can meet this particular objection by incorporating into the system of his paper one of the standard devices for avoiding the epistemological paradoxes. (Church, forthcoming).

w_1. Then, $w_1 \models p \mathrel{\Box\!\!\rightarrow} (q \vee r)$ but $w_1 \not\models (p \mathrel{\Box\!\!\rightarrow} q) \vee (p \mathrel{\Box\!\!\rightarrow} r)$. The inference is of course valid for necessary q's and r's.

[9]See also Dummett (2007a, pp. 303–4).

[10]See e.g. Dummett (1994, p. 296).

The idea has been recently developed by Bernard Linsky and Alexander Paseau.[11] Though the Church-Fitch proof makes no use of self-referential sentences, Linsky and Paseau observe, it is nevertheless invalid on a logical account of knowledge reminiscent of Russell's theory of types. The account rests on the following two rules:

(16) If φ has no occurrences of $\mathcal{K}$, φ is of type 0 (φ_0);
(17) If φ is of type n, then $\mathcal{K}\varphi$ is of type $n+1$ ($\mathcal{K}_{n+1}\varphi_n$).

The intuitive idea is that formulae are assigned *logical types*, where types reflect the nesting of occurrences of $\mathcal{K}$ within each formula. On this view, the above argument (1)—(6) is invalid. For let p be of type 0. Then, the argument is blocked at line 3:

1	(1*)	$\mathcal{K}_2(p_0 \wedge \neg\mathcal{K}_1 p_0)$	A
1	(2*)	$\mathcal{K}_2 p_0 \wedge \mathcal{K}_2\neg\mathcal{K}_1 p_0$	1*, DIST
1	(3*)	$\mathcal{K}_2 p_0 \wedge \neg\mathcal{K}_1 p_0$	2*, FACT

Unless it is assumed that $\mathcal{K}_{n+1}\varphi_n \rightarrow \mathcal{K}_n\varphi_n$, line (3*) is not necessarily a contradiction.

Does the hierarchical treatment represent a viable answer to the Church-Fitch Paradox? And can the mere appeal to intuitionistic logic salvage semantic anti-realism from its paradoxical consequences?

4 The Paradox of Idealization

There is a dispute among anti-realists over whether or not knowability requires idealization. Strict Finitists think that idealization is not required. The word 'knowable', for them, is to be interpreted as 'possibly known by agents just like us'. As one might expect, though, Strict Finitism carries highly revisionary consequences. On that view, any decidable proposition that cannot be known for mere 'medical' limitations—e.g. any arithmetical proposition involving very large numbers—turns out to be 'meaningless', if not false. But this result is hardly acceptable. As Dummett puts it:

> The intuitionist sanctions the assertion, for any natural number, however large, that it is either prime or composite, since we have

[11] See Linsky (forthcoming) and Paseau (ms.).

> a method that will, at least in principle, decide the question. But suppose that we do not, and perhaps in practice cannot apply that method: is there nevertheless a fact of the matter concerning whether the number is prime or not? There is a strong impulse that there must be. (Dummett, 1975, pp. 296-70)

Most anti-realists thus concede that 'knowable' in WVER is to be read as 'knowable *in principle*'—i.e. knowable by agents endowed with cognitive capacities like ours or that finitely exceed ours.[12] Thus Tennant:

> The truth does not have to be knowable by all and sundry, regardless of their competence to judge. [...] This would be to hostage too much of what is true to individual misfortune. At the very least, we have to abstract or idealize away from the limitations of actual individuals. [...] At the very least, then, we have to imagine that we can appeal to an ideal cognitive representative of our species". (Tennant, 1997, p. 144)

Call such anti-realists *moderate* anti-realists. In spite of its initial plausibility, we argue, this move runs the risk of becoming a Trojan horse.

Our argument starts from the moderate anti-realist's concession that there are feasibly unknowable truths—i.e. truths that because of their length, or of the length of their justification or proof, as the case may be, can only be known by agents whose cognitive capacities finitely exceed ours. Although similar truths can be both empirical and mathematical, we focus on the mathematical case for ease of exposition, here and throughout. Let q be one such truth and let 'Ix' read 'x is an ideal agent', where an agent counts as ideal if and only if her cognitive capacities—perceptual discrimination, memory, working memory etc.[13]—finitely exceed ours.[14] We then interpret the anti-realist's concession as meaning that, of necessity, any agent who knows q is an ideal agent. In symbols:

$$(18)\ \Box\, \forall x(\mathcal{K}_x q \rightarrow Ix).$$

[12] See Dummett (1975) and, especially, Tennant (1997, Chapter 5).

[13] These capacities can presumably be assigned numerical values, so that no essential reference to actual beings needs to be made. We could select a given range of values R for each capacity—the values that natural epistemic agents typically get—and then define an agent as ideal if the value assigned to at least one of her capacities is greater than every value in R.

[14] We shall consider an alternative definition of an idealized agent in Section 5 below.

It is now but a short step to prove that the conjunction $q \wedge \neg\exists x Ix$ is unknowable. For knowledge of its first conjunct obtains only if there is an ideal agent, which contradicts what the second conjunct says, namely that there are no ideal agents. We first present a semantic proof and then regiment it as a natural deduction.

> *Proof*: Assume that $q \wedge \neg\exists x Ix$ is knowable. Then there is a world w where some agent knows $q \wedge \neg\exists x Ix$. Call this agent a. By (18) every agent who knows q in w is ideal. Therefore, a is ideal. However, since a knows $q \wedge \neg\exists x Ix$, by DIST and FACT, $\neg\exists x Ix$ is true at w. Hence, a cannot be an ideal agent. Contradiction. Therefore, $q \wedge \neg\exists x Ix$ is unknowable. ■

For our regimentation we use the following rule valid in K, which we label MR:

X	(i)	$\Box\varphi$	...
	$\vdots$		
Y	(j)	$\Diamond\psi$	...
	$\vdots$		
k	(k)	$\varphi \wedge \psi$	A
	$\vdots$		
k	(l)	$\bot$	...
	$\vdots$		
$X \cup Y$	(m)	$\bot$	i, j, k–l (MR)

We can then represent the argument as follows:

19	(19)	$\exists\varphi(\varphi \wedge \Box\forall x(\mathcal{K}_x\varphi \rightarrow Ix))$	A
20	(20)	$\neg\exists xIx$	A
21	(21)	$\forall\varphi(\varphi \rightarrow \Diamond\exists y\mathcal{K}_y\varphi)$	WVER
22	(22)	$q \wedge \Box\forall x(\mathcal{K}_xq \rightarrow Ix))$	A
22	(23)	$\Box\forall x(\mathcal{K}_xq \rightarrow Ix)$	22, $\wedge$-E
22	(24)	q	22, $\wedge$-E
22, 20	(25)	$q \wedge \neg\exists xIx$	20,24, $\wedge$-I
21	(26)	$(q \wedge \neg\exists xIx) \rightarrow \Diamond\exists y\mathcal{K}_y(q \wedge \neg\exists xIx)$	21, $\forall$-E
20, 21, 22	(27)	$\Diamond\mathcal{K}(q \wedge \neg\exists xIx)$	25,26, $\rightarrow$-E
28	(28)	$\forall x(\mathcal{K}_xq \rightarrow Ix) \ \wedge \ \exists y\mathcal{K}_y(q \wedge \neg\exists xIx)$	A
28	(29)	$\exists y\mathcal{K}_y(q \wedge \neg\exists xIx)$	28, $\wedge$-E
28	(30)	$\exists y\mathcal{K}_yq \wedge \exists y\mathcal{K}_y(\neg\exists xIx)$	29, DIST
28	(31)	$\exists y\mathcal{K}_yq$	30, $\wedge$-E
28	(32)	$\forall x(\mathcal{K}_xq \rightarrow Ix)$	28, $\wedge$-E
33	(33)	$\mathcal{K}_aq$	A
28	(34)	$\mathcal{K}_aq \rightarrow Ia$	32, $\forall$-E
28, 33	(35)	Ia	33,34, $\rightarrow$-E
28, 33	(36)	$\exists xIx$	35, $\exists$-I
28	(37)	$\exists y\mathcal{K}_y(\neg\exists xIx)$	30, $\wedge$-E
28	(38)	$\neg\exists xIx$	37, FACT
28, 33	(39)	$\bot$	36,38, $\wedge$-I
28	(40)	$\bot$	31,33,39, $\exists$-E
20, 21, 22	(41)	$\bot$	23,27,28–40, MR
19, 20, 21	(43)	$\bot$	19,22,41, $\exists$-E

The argument generalizes. Similar proofs can be constructed for every formula φ and $\mathcal{P}(x,\varphi)$ such that both of the following hold:

$$(19') \ \varphi \wedge \Box \ \forall x(\mathcal{K}_x\varphi \rightarrow \mathcal{P}(x,\varphi));$$
$$(20') \ \neg\exists x\mathcal{P}(x,\varphi).$$

Relevant instances of $\mathcal{P}(x,\varphi)$ may include traditional necessary conditions for knowledge, such as justification or belief. The Knowability Paradox itself may be thought of as an instance of our paradox, starting from $\mathcal{P}(x,\varphi) \equiv \mathcal{K}_x\varphi$ and the following trivial instance of (19′):

$$(19'') \ \varphi \wedge \Box \ \forall x(\mathcal{K}_x\varphi \rightarrow \mathcal{K}_x\varphi).$$

We take our proof to be problematic for moderate anti-realists, in particular for those who appeal to either intuitionistic logic or knowledge types as a means of blocking the Paradox of Knowability. To begin with, the new proof appears to resist basic hierarchical treatments of the kind suggested by Linsky and Paseau. For although the definition of 'Ix' involves reference to cognitive capacities, it does not involve reference to knowledge of any particular proposition. Therefore, typing '$\mathcal{K}$' does not seem to get to the heart of the matter, unless anti-realists have good reasons to also type 'Ix', as well as any other predicates that may be substituted in (19′) and (20′). More importantly, the argument is intuitionistically valid, and even intuitionists seem forced to accept its main premise, *viz.* that there are feasibly unknowable truths. For let q be some decidable yet undecided mathematical statement whose decision procedure is feasibly unperformable—say some statement to the effect that m is prime, where m is some very large natural number. Then, q satisfies both of the following:

$$(44)\ q \to \Box\forall x(\mathcal{K}_x q \to Ix);$$
$$(45)\ \neg q \to \Box\forall x(\mathcal{K}_x \neg q \to Ix).$$

Since however q is *ex hypothesis* decidable, the law of excluded middle applies to it. The existence of a feasibly unknwable truth can then be easily derived from $q \vee \neg q$, (44), and (45).[15]

This highlights a striking asymmetry between the original Paradox of Knowability and the Paradox of Idealization. Whereas the former argument proceeds from the existence of empirical forever unknown truths, the latter paradox only requires that there be feasibly unknowable *mathematical* truths. Yet, whilst the existence of empirical forever unknown truths may be intuitionistically questionable, the existence of mathematical feasibly unknowable truths seems unexceptionable, even by Dummett's enforced intuitionistic standards.

If the argument is not to be regarded as a *reductio* of WVER, anti-realists who are willing to solve the problem have no choice but to deny either (19)

[15] *Proof*: Assume $q \vee \neg q$, (44) and (45) for disjunction elimination. Then assume q for arrow introduction. Now show that $\Box\forall x(\mathcal{K}_x q \to Ix)$ follows from (44) by arrow elimination. By conjunction introduction, we can then infer $q \wedge \Box\forall x(\mathcal{K}_x q \to Ix)$, from which $\exists\varphi(\varphi \wedge \Box\forall x(\mathcal{K}_x \varphi \to Ix))$ can be derived by existential introduction. By arrow introduction, it follows that $q \to \exists\varphi(\varphi \wedge \Box\forall x(\mathcal{K}_x \varphi \to Ix))$. By similar reasoning, we can derive that $\neg q \to \exists\varphi(\varphi \wedge \Box\forall x(\mathcal{K}_x \varphi \to Ix))$. By disjunction elimination, it follows that $\exists\varphi(\varphi \wedge \Box\forall x(\mathcal{K}_x \varphi \to Ix))$. ■

or (20). We shall argue in what follows that neither option seem viable, irrespectively of whether intuitionistic logic is adopted.

5 Objections and Replies

We now turn to some potential concerns about the soundness of our proof. Let us begin with (19), i.e. the claim that there are feasibly unknowable truths. In light of the Paradox of Idealization, anti-realists might reconsider their moderation and argue that for any proposition φ, it is possible that φ be known by a non-ideal agent:

$$(44)\ \forall\varphi(\varphi \to \Diamond\exists x(\mathcal{K}_x\varphi \wedge \neg Ix)).$$

Since (44) intuitionistically entails the falsity of (19), our paradox would be blocked. The thought might be motivated in a number of ways. For instance, anti-realists might be committed to the claim, quite innocent for them, that, if there is a method to verify φ, then there is a possible world whose space-time structure is such that agents with cognitive capacities just like ours know that φ. Alternatively, they might claim that for any true φ, there is a possible world at which φ itself, or its proof, are expressed in a language that renders them cognitively accessible.[16]

This kind of objection is problematic on two major counts. To begin with, it is not at all clear that the envisaged agents would not be ideal. Plausibly, a different space-time structure, or a radically different language, would make φ feasibly knowable by carrying an improvement of the agents' cognitive capacities. But then, pending a more accurate description of the world in question, it is open to argue that those agents *are* after all ideal. More importantly, even we conceded that those agents are not ideal, a variant of our paradox shows that similar objections do not really get off the ground. For let S be a description of the space-time structure of the actual world, or of the totality of the languages that are actually used. Then, consider the modified premise:

$$(19^*)\ \exists\varphi((\varphi \wedge S) \wedge \Box\, \forall x(\mathcal{K}_x(\varphi \wedge S) \to Ix).$$

In perfect analogy with the Paradox of Idealization, we can argue as follows:

[16] We thank Cesare Cozzo and Luca Incurvati for having pressed this point.

Proof: Assume that $(q \wedge S) \wedge \neg\exists x Ix$ is knowable. Then there is a world w where some agent a knows $(q \wedge S) \wedge \neg\exists x Ix$. This forces w to have the space-time structure described by S. It also follows that $\neg\exists x Ix$ is true in w. Therefore, a is a non-idealized knower of q in a world whose space-time structure is S. Contradiction, since we are assuming that, necessarily, $\forall x(\mathcal{K}_x(q \wedge S) \rightarrow Ix)$. Thus, $(q \wedge S) \wedge \neg\exists x Ix$ is unknowable. ■

Anti-realists could perhaps reply by availing themselves of the characteristic weakness of intuitionistic logic. They could deny (19), on the one hand, and express their moderation by claiming that not every truth is feasibly knowable, on the other:

$$(46)\ \neg\forall\varphi(\varphi \rightarrow \Diamond\exists x(\mathcal{K}_x\varphi \wedge \neg Ix)).$$

Classically, (46) and the denial of (19) contradict, but not intuitionistically.[17] The problem with this, of course, is that intuitionists can *prove* the existence of feasibly unknowable truths, unless they can show that the argument we gave at the end of Section 4 is intuitionistically lacking.

Moderate anti-realists might bite the bullet and deny (20) instead. But would this be advisable? We see two possibilities, depending on the epistemic status of $\neg\exists x Ix$. If an agent counts as ideal just in case her cognitive capacities finitely exceed those of *any actual epistemic agent*, $\neg\exists x Ix$ is indeed an a priori truth. It would say that there are no (actual) epistemic agents whose cognitive capacities finitely exceed those of any (actual) epistemic agent, which is of course a truism. On the other hand, if anti-realists took $\neg\exists x Ix$ to be empirical, perhaps defining—as Tennant might be taken as suggesting—'Ix' in terms of the average human cognitive capacities, they would be denying it as a mere consequence of their moderation and of their acceptance of WVER. Yet, on these assumptions, the anti-realist's moderation would result from considerations concerning *our* congnitive capacities, and the truth of an empirical matter should not be decided by those considerations together with a metaphysical assumption such as WVER.

Be that as it may, even if anti-realists went as far denying $\neg\exists x Ix$, this would not help them with another variant of our paradox resting on the following assumption:

[17] Notice that this move just is the analogous of Williamson's recommendation to accept (12) while denying SVER.

$$(47)\ \Diamond\exists\varphi(\varphi \wedge \Box\ \forall x(\mathcal{K}_x\varphi \rightarrow Ix) \wedge \neg\exists x Ix).$$

Presumably, even for an anti-realist there is some feasibly unknowable proposition φ, such that φ and $\neg\exists x Ix$ are co-possible. But now, provided that the relation of accessibility is transitive, or that they accept a strenghtened version of (47),

$$(47')\ \Diamond\exists\varphi(\varphi \wedge \Box\Box\ \forall x(\mathcal{K}_x\varphi \rightarrow Ix) \wedge \neg\exists x Ix),$$

we can run a version of the Paradox of Idealization via either (47) or (47′) and the necessitated formulation of WVER,

$$(\mathsf{WVER}^{**})\ \Box\ \forall\varphi(\varphi \rightarrow \Diamond\mathcal{K}\varphi).$$

Anti-realists could of course reject WVER**, thereby sticking to WVER. This, however, would be a desperate move. For one, it would have an *ad hoc* flavour. For another, it would leave the anti-realist with a contingent version of her core metaphysical tenet. Anti-realists would seem to have only one option left: giving up both (47′) and transitivity. But this would be a surprising consequence of their acceptance of WVER.

Conclusion

The Paradox of Idealization threatens the viability of intuitionist and hierarchical defences of semantic anti-realism. Hierarchical approaches might block the original Knowability Paradox but fail to block the cognate Paradox of Idealization. As for the appeal to intuitionistic logic, it does not help the anti-realist avoid the inconsistency among the three assumptions on which our Paradox depends. Denying (20) does not seem an option, independently of whether classical logic is admitted. Rejecting (19) is tantamount to abandoning moderate anti-realism. Anti-realists who favour either an intuitionist or a hierarchical approach to the Knowability Paradox thus appear to be confronted with a dilemma: they must either negate WVER or give up their moderation. How to cope with this dilemma? Other solutions to the Paradox have been proposed so far.[18] Although they are all controversial, our

[18]See, e.g., Edgington (1985), Tennant (1997) and Tennant (forthcoming)

result suggests that a viable defense of anti-realism turns on whether or not they are acceptable. We leave to anti-realists the hard task of providing an adequate defence of their core metaphysical tenets.

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